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New coherent states for the BDS-Hamiltonian

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In the paper we construct a new set of coherent states for a deformed Hamiltonian of the harmonic oscillator, previously introduced by Beckers, Debergh, and Szafraniec, which we have called the BDS-Hamiltonian. This Hamiltonian depends on the new creation operator a_λ^+, i.e. the usual creation operator displaced with the real quantity λ. In order to construct the coherent states, we use a new measure in the Hilbert space of the Hamiltonian eigenstates, in fact we change the inner product. This ansatz assures that the set of eigenstates be orthonormalized and complete. In the new inner product space the BDS-Hamiltonian is self-adjoint. Using these coherent states, we construct the corresponding density operator and we find the P-distribution function of the unnormalized density operator of the BDS-Hamiltonian. Also, we calculate some thermal averages related to the BDS-oscillators system which obey the quantum canonical distribution conditions.

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1 Introduction

Beginning with the pioneering papers on the one-dimensional harmonic oscillator coherent states (HO-1D-CS) of Glauber, Klauder, Stoler, and Perelomov [1–5], which are based on the use of the properties of the orthodox creation a^+ and annihilation a operators, many papers in which perturbations of these orthodox operators were considered have appeared lately. Such perturbations lead to the new expressions of the harmonic oscillator Hamiltonians and, consequently, to the new coherent states.

Recently, a deformed not self-adjoint Hamiltonian for the harmonic oscillator has been introduced by Beckers, Debergh, and Szafraniec [6]. In a previous paper [7], we have called this Hamiltonian as the *BDS-Hamiltonian* and we have formulated the density matrix approach for this one. The BDS-Hamiltonian

$$H_\lambda = \tfrac{1}{2}\hbar\omega\left[a,\, a_\lambda^+\right]_+ = \tfrac{1}{2}\hbar\omega\left[a,\, a^+\right]_+ + \lambda\hbar\omega a \tag{1}$$

is an analog of the harmonic one-dimensional oscillator Hamiltonian but, instead of the usual creation operator a^+, there appears a new creation operator, displaced

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by the real quantity λ:

$$a_\lambda^+ = a^+ + \lambda I \tag{2}$$

while the corresponding Heisenberg algebra is

$$\left[a,\, a_\lambda^+\right] = \left[a,\, a^+\right] = I. \tag{3}$$

The corresponding Schrödinger stationary equation leads to the following eigenfunctions and eigenvalues [6]:

$$\Psi_{n,\lambda}(x) := \langle x|n;\lambda\rangle = C_n \exp\left(-\frac{m\omega}{2\hbar}x^2\right) H_n\left(\sqrt{\frac{m\omega}{\hbar}}x + \frac{\lambda}{\sqrt{2}}\right), \tag{4}$$

$$E_n = \hbar\omega\left(n + \tfrac{1}{2}\right). \tag{5}$$

Because of the structure of the Hermite polynomials $H_n(\ldots)$, these eigenfunctions are normalizable, but not orthogonal [6]. In order to construct the coherent states related to the BDS-Hamiltonian, it is necessary that these functions must be orthonormalized.

The central idea is the use of a new measure $\mathrm{d}\mu(x) = \mathrm{d}x \, \exp\left[f(x)\right]$, instead of $\mathrm{d}x$:

$$\int_{-\infty}^{+\infty} \mathrm{d}\mu(x)\, |x\rangle\langle x| = 1, \tag{6}$$

$$\begin{aligned} \langle m;\lambda|n;\lambda\rangle &= \int_{-\infty}^{+\infty} \mathrm{d}\mu(x)\, \langle m;\lambda|x\rangle\langle x|n,\lambda\rangle \\ &= \int_{-\infty}^{+\infty} \mathrm{d}x \, \exp\left[f(x)\right] \Psi^*_{m,\lambda}(x)\Psi_{n,\lambda}(x). \end{aligned} \tag{7}$$

In other words, we have changed the inner product.

Using the orthogonality properties of the Hermite polynomials, we find that the new measure must be

$$\mathrm{d}\mu(x) = \mathrm{d}x \, \exp\left(-\frac{\lambda^2}{2} - \lambda\sqrt{2}\sqrt{\frac{m\omega}{\hbar}}x\right), \tag{8}$$

i.e. a new weight function. This function assures that the eigenfunctions of the BDS-Hamiltonian form a complete orthonormal set in the space of the square integrable functions on the real axis, with respect to the new measure (8). So, in the new inner product space, the BDS-Hamiltonian is self-adjoint. Therefore, the set of the eigenstates (4), with the normalization constant

$$C_n = \left[\frac{1}{2^n n!}\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\right]^{1/2} \tag{9}$$

is a complete, orthonormal basis in Hilbert space:

$$\langle m;\lambda|n;\lambda\rangle = \delta_{mn}\,, \tag{10}$$

i.e. the space of the square integrable functions on the $(-\infty, +\infty)$, with respect to the measure $\mathrm{d}\mu(x)$.

The aim of this paper is to construct the coherent states of the BDS-Hamiltonian. This will be carried out in Section 2. Section 3 is dedicated to the construction of the corresponding density operator in the coherent states representation for a more general chaotic state, i.e. for the thermal state. In Section 4 we solve the Bloch equation for the BDS-Hamiltonian, as an alternative way to obtain the expression of the unnormalized density operator. Using this expression, we calculate the partition function $Z_\lambda(\beta)$ for a quantum system of the BDS-oscillators which obeys the conditions of the quantum canonical distribution. Section 5 is reserved for the diagonal representation of the canonical (or unnormalized) density operator. By means of this diagonal representation, in Section 6 we calculate the thermal averages of some normal ordered operators (particle number operator, BDS-Hamiltonian, entropy), the dispersion of some operators (and, as an application, the position-momentum uncertainty product) and the degree of coherence. Finally, we make some concluding remarks.

2 Coherent states

By using the well-known expressions for the orthodox bosonic annihilation and creation operators [8]

$$a = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}x + \mathrm{i}\frac{1}{\sqrt{m\hbar\omega}}p\right) = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}x + \sqrt{\frac{\hbar}{m\omega}}\frac{\partial}{\partial x}\right), \tag{11}$$

$$a^+ = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}x - \mathrm{i}\frac{1}{\sqrt{m\hbar\omega}}p\right) = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}x - \sqrt{\frac{\hbar}{m\omega}}\frac{\partial}{\partial x}\right), \tag{12}$$

it is not difficult to prove that the new creation a_λ^+ and orthodox annihilation a operators, acting on the basis vectors in the Hilbert space $|n;\lambda\rangle$, lead to the following relations:

$$a_\lambda^+|n;\lambda\rangle = \sqrt{n+1}\,|n+1;\lambda\rangle, \tag{13}$$

$$a|n;\lambda\rangle = \sqrt{n}\,|n-1;\lambda\rangle, \tag{14}$$

i.e. the same relations as for the usual Fock-vectors $|n\rangle \equiv |n;0\rangle$ for the orthodox one-dimensional harmonic oscillator (HO-1D). We can say that the vectors $|n;\lambda\rangle$ are different families of the BDS-states depending on a real parameter λ, which overlap the basis Fock states $|n\rangle$. Due to the correspondence $E_n \longrightarrow |n;\lambda\rangle$, the parameter λ plays the role of the degeneracy parameter, the energy levels of the BDS-Hamiltonian having an infinite degeneration degree. Consequently, the new number operator N_λ fulfils the equation

$$N_\lambda|n;\lambda\rangle := a_\lambda^+ a|n;\lambda\rangle = n|n;\lambda\rangle. \tag{15}$$

So, the vectors $|n;\lambda\rangle$ are eigenvectors of the number operator.

On the other hand, by the successive application of the operator a_λ^+ on the vacuum state $|0;\lambda\rangle$, we are led to the relation

$$|n;\lambda\rangle = \frac{1}{\sqrt{n!}} \left(a_\lambda^+\right)^n |0;\lambda\rangle. \tag{16}$$

We define a new family of coherent states (CS) $|z;\lambda)$ of the BDS-Hamiltonian, in the usual way (as the eigenstate of the annihilation operator [1], [2]):

$$a|z;\lambda) = z|z;\lambda) \tag{17}$$

and we name these states as the *BDS-coherent states* (BDS-CS). We have used for the BDS-CS the notation $|z;\lambda)$ and not the usual notation for coherent states $|z;\lambda\rangle$, in order to avoid confusion with the number vectors $|n;\lambda\rangle$ in the case of natural values for the variable z. Then, the corresponding relation of the orthodox creation operator a^+ will be

$$(z;\lambda|a^+ = z^*(z;\lambda|, \tag{18}$$

where z^* is the complex conjugate of the complex z-number.

The development of the BDS-CS with respect to the number states is

$$|z;\lambda) = \sum_{n=0}^{\infty} \langle n;\lambda|z;\lambda)\, |n;\lambda\rangle. \tag{19}$$

From Eq. (17) it follows that

$$\langle n;\lambda|a|z;\lambda) = z\langle n;\lambda|z;\lambda). \tag{20}$$

The left hand side of this relation can be transformed successively as follows:

$$\langle n;\lambda|(a|z;\lambda)) = (\langle n;\lambda|a^+)|z;\lambda) = (\langle n;\lambda|a_\lambda^+)|z;\lambda) - \lambda\langle n;\lambda|z;\lambda) \tag{21}$$

and using the properties of the operators a_λ^+ and a and the orthonormalization relation of the basis vectors $|n;\lambda\rangle$, after straightforward calculations, we obtain the recursion relation

$$\langle n;\lambda|z;\lambda) = \frac{(z+\lambda)^n}{\sqrt{n!}} \langle 0;\lambda|z;\lambda). \tag{22}$$

Then, the expression for the coherent state becomes

$$|z;\lambda) = \langle 0;\lambda|z;\lambda) \sum_{n=0}^{\infty} \frac{(z+\lambda)^n}{\sqrt{n!}}\, |n;\lambda\rangle. \tag{23}$$

Imposing that the BDS-CS be normalized to unity and using the orthogonality relation (10), we obtain the value of the normalization constant:

$$\langle 0;\lambda|z;\lambda) = (z;\lambda|0;\lambda\rangle = \exp\left(-\tfrac{1}{2}|z+\lambda|^2\right). \tag{24}$$

Then, the final expression for the BDS-CS becomes:

$$|z;\lambda) = \exp\left(-\tfrac{1}{2}|z+\lambda|^2\right)\sum_{n=0}^{\infty}\frac{(z+\lambda)^n}{\sqrt{n!}}\,|n;\lambda\rangle. \tag{25}$$

It is easy to prove that these new coherent states own all properties required for the coherent states, i.e. [9]:

a) Two BDS-CS are never orthogonal to each other, since:

$$(z;\lambda|z';\lambda) = \exp\left[-\tfrac{1}{2}|z+\lambda|^2 - \tfrac{1}{2}|z'+\lambda|^2 + (z^*+\lambda)(z'+\lambda)\right]. \tag{26}$$

b) The BDS-CS determine a resolution of the identity:

$$\int_C \frac{\mathrm{d}^2 z}{\pi}\,|z;\lambda)(z;\lambda| = I, \tag{27}$$

where $\mathrm{d}^2 z = \mathrm{d}(\mathrm{Re}\, z)\,\mathrm{d}(\mathrm{Im}\, z) = r\,\mathrm{d}r\,\mathrm{d}\varphi$, since $z = r\exp(\mathrm{i}\varphi)$. So, the BDS-CS form an overcomplete set of states.

c) There exists a function $K(z,z') := (z;\lambda|z';\lambda)$ called the reproducing kernel, because it satisfies the relation

$$K(z,z') = \int_C \frac{\mathrm{d}^2 z''}{\pi}\,K(z,z'')K(z'',z'). \tag{28}$$

d) The BDS-CS have remarkable functional analytic and meromorphic properties. Let $|\Psi\rangle$ be an arbitrary vector in the Hilbert space. Writing $\Psi(z;\lambda) := (z;\lambda|\Psi\rangle$, from Eq. (23) we obtain that the reproducing property of the $K(z,z')$ implies that

$$\Psi(z';\lambda) = \int_C \frac{\mathrm{d}^2 z}{\pi} K(z',z)\Psi(z;\lambda). \tag{29}$$

On the other hand, by performing the complex conjugate of Eq. (19) we obtain

$$\Psi(z;\lambda) = \exp\left(-\tfrac{1}{2}|z+\lambda|^2\right)\sum_{n=0}^{\infty}\frac{(z^*+\lambda)^n}{\sqrt{n!}}\,\langle n;\lambda|\Psi\rangle := \exp\left(-\tfrac{1}{2}|z+\lambda|^2\right) f(z^*+\lambda), \tag{30}$$

where $f(z^*+\lambda)$ is an entire analytic function of the complex variable $z^*+\lambda$. It follows that the correspondence $|\Psi\rangle \longrightarrow \Psi(z;\lambda)$ is continuous for each $z \in C$.

e) When we write

$$a = \frac{1}{\sqrt{2}}(Q+\mathrm{i}P); \qquad\qquad z = \frac{1}{\sqrt{2}}(q+\mathrm{i}p), \tag{31}$$

the real numbers q and p can be identified as the eigenvalues of the position Q and momentum P operators. The last relation allows the identification of the complex z-plane with the phase space of the system, so that BDS-CS $|z;\lambda) := |p,q;\lambda)$ is in fact a phase space realization of the quantum mechanics [10], with λ as real parameter.

The following relation exists:

$$\lim_{\lambda\to 0} F_\lambda = F_0, \tag{32}$$

where F_λ is a quantity or formula related with the BDS-Hamiltonian H_λ and F_0 is the corresponding quantity or formula related with the usual HO-1D Hamiltonian H_0.

3 Density matrix in thermodynamic equilibrium

We consider a single-mode free radiation field of BDS-oscillators with the angular frequency ω in thermodynamic equilibrium to a bigger system (a "reservoir" or a "bath"), at the temperature T, which obeys the conditions of the quantum canonical distribution. The more general chaotic state of this system is characterized by unnormalized density operator [8]

$$\rho_\lambda^c = \left(\frac{\overline{n}}{\overline{n}+1}\right)^{1/2} \sum_{n=0}^{\infty} \left(\frac{\overline{n}}{\overline{n}+1}\right)^n |n;\lambda\rangle\langle n;\lambda|, \tag{33}$$

with the mean occupancy

$$\overline{n} = \left[\exp\left(\frac{\hbar\omega}{k_{\mathrm{B}}T}\right) - 1\right]^{-1}. \tag{34}$$

The unnormalized density operator is diagonal in the number state representation:

$$\langle m;\lambda|\rho_\lambda^c|n;\lambda\rangle = \left(\frac{\overline{n}}{\overline{n}+1}\right)^{1/2} \left(\frac{\overline{n}}{\overline{n}+1}\right)^n \delta_{mn}. \tag{35}$$

In the BDS-CS representation, the non-diagonal elements of the unnormalized density matrix are

$$(z;\lambda|\rho_\lambda^c|z';\lambda) = \left(\frac{\overline{n}}{\overline{n}+1}\right)^{1/2} \exp\left[-\tfrac{1}{2}|z+\lambda|^2 - \tfrac{1}{2}|z'+\lambda|^2 + \frac{\overline{n}}{\overline{n}+1}(z^*+\lambda)(z'+\lambda)\right]. \tag{36}$$

On the other hand, the partition function is obtainable as the trace of the unnormalized density operator

$$Z_\lambda(\beta) = \mathrm{Tr}\,\rho_\lambda^c = \int_C \frac{\mathrm{d}^2 z}{\pi}\,(z;\lambda|\rho_\lambda^c|z;\lambda) = [\overline{n}(\overline{n}+1)]^{1/2} = \frac{1}{2\sinh\left(\frac{1}{2}\beta\hbar\omega\right)}\,, \tag{37}$$

where β is the well-known temperature variable $\beta = (k_{\mathrm{B}}T)^{-1}$. The above expression is the same as for the HO-1D. This result provides that all thermodynamical and statistical properties of the BDS-oscillator system are the same as for the HO-1D system.

The previous integral is of the following kind [11], [12]:

$$\int_C \frac{\mathrm{d}^2 z}{\pi} \exp\left[-a|z|^2 + bz + cz^* + ez^2 + f(z^*)^2\right] = \frac{1}{\sqrt{a^2 - 4ef}} \exp\left(\frac{abc + b^2 f + c^2 e}{a^2 - 4ef}\right), \tag{38}$$

where the integral is convergent for $\mathrm{Re}\, a > |e^* + f|$ while b and c may be arbitrary.

4 Bloch equation

It is well known from the quantum mechanical books (see, e.g., [8] and [12]) that the unnormalized density operator ρ_λ^c obeys the Bloch equation [13]:

$$-\frac{\partial}{\partial \beta} \rho_\lambda^c = H_\lambda \rho_\lambda^c\,; \qquad\qquad \lim_{\beta \to 0} \rho_\lambda^c = 1. \tag{39}$$

In the coherent state representation, the Bloch equation is

$$-\frac{\partial}{\partial \beta} (z;\lambda|\rho_\lambda^c|z';\lambda) = (z;\lambda|H_\lambda \rho_\lambda^c|z';\lambda), \tag{40}$$

with the boundary condition (when $\beta \to 0$, then $\overline{n} \to \infty$)

$$\lim_{\beta \to 0} (z;\lambda|\rho_\lambda^c|z';\lambda) = (z;\lambda|z';\lambda)\,. \tag{41}$$

This problem was first solved by Vakarchuk [14]. It is necessary to repeat here briefly these calculations, only in order to point out three steps. Firstly, instead of the function $(z;\lambda|\rho_\lambda^c|z';\lambda)$ (which is not an analytical function) it is useful to introduce a complex-valued entire function of two complex variables z^* and z' (the so-called Glauber's function [1]):

$$R_\lambda(z^*, z'; \beta) := \frac{(z;\lambda|\rho_\lambda^c|z';\lambda)}{\langle 0;\lambda|z;\lambda)(z';\lambda|0;\lambda\rangle}\,. \tag{42}$$

Secondly, it is proved (see Ref. [14]) that Glauber's function is the solution of the following Bloch equation:

$$-\frac{\partial}{\partial \beta} R_\lambda(z^*, z'; \beta) = H_\lambda \left(z^*, \frac{\partial}{\partial z^*}\right) R_\lambda(z^*, z'; \beta). \tag{43}$$

Consequently, the boundary condition is written as

$$\lim_{\beta \to 0} R_\lambda(z^*, z'; \beta) = \exp\left[(z^* + \lambda)(z' + \lambda)\right]. \tag{44}$$

In the right hand side of (43), there appears the operator $H_\lambda(z^*, \partial/\partial z^*)$,which results from the BDS-Hamiltonian by substituting for each orthodox creation operator a^+ the variable z^* and for each annihilation operator a the first derivative with respect to the variable z^*, i.e. $\partial/\partial z^*$. This fact is a consequence of the validity of the canonical commutation relation (3) and can be performed if and only if the Hamiltonian is a normally ordered function with respect to the operators a^+ and a.

The next step is the functional change

$$R_\lambda(z^*, z'; \beta) = \exp\left[G_\lambda(z^*, z'; \beta)\right] \tag{45}$$

and, consequently, the Bloch equation becomes

$$-\frac{\partial}{\partial\beta} G_\lambda(z^*, z'; \beta) = H_\lambda\left(z^*, \frac{\partial G_\lambda}{\partial z^*}\right), \tag{46}$$

with the boundary condition for the new introduced function

$$\lim_{\beta\to 0} G_\lambda(z^*, z'; \beta) = (z^* + \lambda)(z' + \lambda). \tag{47}$$

Now, the problem is to find the function $G_\lambda(z^*, z'; \beta)$, that is to solve the last differential equation with the corresponding boundary condition. This equation is of the same kind as the Hamilton–Jakobi equation from the analytical mechanics.

For the BDS-Hamiltonian, this equation becomes

$$-\frac{\partial}{\partial\beta} G_\lambda(z^*, z'; \beta) = \hbar\omega(z^* + \lambda)\frac{\partial}{\partial z^*} G_\lambda(z^*, z'; \beta) + \frac{\hbar\omega}{2}. \tag{48}$$

By a simple functional change, in order to eliminate the constant term from r.h.s., we obtain a differential equation with separable variables. After straightforward calculations, the solution can be written as follows:

$$G_\lambda(z^*, z'; \beta) = -\beta\tfrac{1}{2}\hbar\omega + (z^* + \lambda)(z' + \lambda)\,\mathrm{e}^{-\beta\hbar\omega}, \tag{49}$$

where we have imposed the symmetry condition of the function $G_\lambda(z^*, z'; \beta)$ with respect to the variables z^* and z', like the same property of the "mother" function $(z; \lambda|\rho^c_\lambda|z'; \lambda)$. Consequently, using Eqs. (41), (44) and (34), this equation becomes

$$(z; \lambda|\rho^c_\lambda|z'; \lambda) = \left(\frac{\overline{n}}{\overline{n}+1}\right)^{1/2} \exp\left[-\tfrac{1}{2}|z+\lambda|^2 - \tfrac{1}{2}|z'+\lambda|^2 + \frac{\overline{n}}{\overline{n}+1}(z^*+\lambda)(z'+\lambda)\right]. \tag{50}$$

This relation shows that it is possible to obtain the unnormalized density operator in the BDS-CS representation also by solving the Bloch equation. This is due to the relatively simple structure of the BDS-Hamiltonian in comparison with other potentials for which this approach is very difficult or impossible.

As a consequence of the fact that the partition function is independent of the displacement parameter λ, all thermodynamical and statistical properties of the BDS-quantum system are the same as for the HO-1D-system.

Based on the previous results, at the end of this section we point out that the connection between the normalized ρ_λ and the unnormalized ρ^c_λ density operator is

$$\rho_\lambda = \frac{1}{Z_\lambda(\beta)}\rho^c_\lambda = \frac{1}{[\overline{n}(\overline{n}+1)]^{\frac{1}{2}}}\rho^c_\lambda, \tag{51}$$

while the corresponding relation for the normalized density matrix in BDS-CS representation is

$$(z;\lambda|\rho_\lambda|z';\lambda) = \frac{1}{\overline{n}+1}\exp\left[-\frac{1}{2}|z+\lambda|^2 - \frac{1}{2}|z'+\lambda|^2 + \frac{\overline{n}}{\overline{n}+1}(z^*+\lambda)(z'+\lambda)\right]. \tag{52}$$

5 Diagonal representation

It is well known (Refs. [4], [15]) that in quantum optics and, more specifically, in the formalism of coherent states, the density operator can be mapped onto two distinct distribution functions: the Q-distribution or covariant form and the P-distribution or contravariant form. The first distribution is a non-negative function, while the P-distribution can assume negative values.

The P-function or the contravariant form of the unnormalized density operator $\rho_\lambda^{\mathrm{c}}$ is defined by the following operator equality [3]:

$$\rho_\lambda^{\mathrm{c}} = \int_C \frac{\mathrm{d}^2 z}{\pi}\,|z;\lambda)P_\lambda(z^*,z;\beta)(z;\lambda|. \tag{53}$$

This is the so-called diagonal representation of the density operator in terms of the BDS-CS. The function $P_\lambda(z^*,z;\beta)$ is called the P-distribution function. In order to obtain the P-distribution function for the BDS-Hamiltonian, we observe that this function has a Gaussian form [11] and we try to find this function in the following form:

$$P_\lambda(z^*,z;\beta) = C_{\mathrm{N}}\exp\left[-q|z+\lambda|^2 + s(z+\lambda) + t(z^*+\lambda)\right], \tag{54}$$

where q is a real positive constant, s, and t are complex constants, and C_{N}, q, s, and t must be determined.

We calculate the non-diagonal elements of the unnormalized density matrix:

$$(\sigma;\lambda|\rho_\lambda^{\mathrm{c}}|\sigma';\lambda) = \int_C \frac{\mathrm{d}^2 z}{\pi}\,(\sigma;\lambda|z;\lambda)P_\lambda(z^*,z;\beta)(z;\lambda|\sigma';\lambda), \tag{55}$$

where σ and σ' are complex variables. Using Eqs. (51), (49), (26), we obtain:

$$\left(\frac{\overline{n}}{\overline{n}+1}\right)^{1/2}\exp\left[\frac{\overline{n}}{\overline{n}+1}(\sigma^*+\lambda)(\sigma'+\lambda)\right] =$$
$$= C_{\mathrm{N}}\int_C \frac{\mathrm{d}^2 z}{\pi}\exp\left[-(q+1)|z+\lambda|^2 + (\sigma^*+\lambda+s)(z+\lambda) + (\sigma'+\lambda+t)(z^*+\lambda)\right]. \tag{56}$$

By using Eq. (38) once again, we find that we must make the following identification:

$$C_N = \left(\frac{\overline{n}+1}{\overline{n}}\right)^{1/2}; \qquad q = \frac{1}{\overline{n}}; \qquad s = t = 0. \tag{57}$$

Finally, after these straightforward calculations, we obtain the P-distribution function for the BDS-Hamiltonian:

$$P_\lambda(z^*, z; \beta) = \left(\frac{\overline{n}+1}{\overline{n}}\right)^{1/2} \exp\left(-\frac{1}{\overline{n}}|z+\lambda|^2\right) \tag{58}$$

and that the diagonal representation of the unnormalized density operator becomes

$$\rho_\lambda^c = \left(\frac{\overline{n}+1}{\overline{n}}\right)^{1/2} \int_C \frac{\mathrm{d}^2 z}{\pi} \, |z;\lambda) \exp\left(-\frac{1}{\overline{n}}|z+\lambda|^2\right)(z;\lambda|. \tag{59}$$

Its trace is, of course, the partition function

$$Z_\lambda(\beta) = \int_C \frac{\mathrm{d}^2\sigma}{\pi} \, (\sigma;\lambda|\rho_\lambda^c|\sigma;\lambda) = \left(\frac{\overline{n}+1}{\overline{n}}\right)^{1/2}$$
$$\times \int_C \frac{\mathrm{d}^2 z}{\pi} \exp\left(-\frac{1}{\overline{n}}|z+\lambda|^2\right)(z;\lambda| \left[\int_C \frac{\mathrm{d}^2\sigma}{\pi} \, |\sigma;\lambda)(\sigma;\lambda|\right] |z;\lambda) = \left[\overline{n}(\overline{n}+1)\right]^{1/2}, \tag{60}$$

where we have used the completeness relation of the BDS-CS. This fact demonstrates that the expression (57) for the diagonal representation of the BDS-unnormalized density operator is correct.

6 Averages for the normal ordered operators

Let us consider an operator $A = A(a_\lambda^+, a)$ in a normal ordered form with respect to the new creation and orthodox annihilation operators, which can be written as the following power series:

$$A = \sum_{m,l=0}^{\infty} A_{ml} \left(a_\lambda^+\right)^m a^l. \tag{61}$$

The thermal average of this operator is

$$\langle A \rangle = \frac{1}{Z_\lambda(\beta)} \int_C \frac{\mathrm{d}^2 z}{\pi} \, (z;\lambda|\rho_\lambda^c A|z;\lambda). \tag{62}$$

Using the completeness relation, we can write:

$$\langle A \rangle = \frac{1}{Z_\lambda(\beta)} \mathrm{Tr}\, \rho_\lambda^c A = \frac{1}{Z_\lambda(\beta)} \int_C \frac{\mathrm{d}^2 z}{\pi} \int_C \frac{\mathrm{d}^2\sigma}{\pi} \, (z;\lambda|\rho_\lambda^c|\sigma;\lambda)(\sigma;\lambda|A|z;\lambda). \tag{63}$$

On the other hand, any operator, particularly the unnormalized density operator, can be decomposed as follows:

$$\rho_\lambda^c = \int_C \frac{\mathrm{d}^2 z}{\pi} \int_C \frac{\mathrm{d}^2\sigma}{\pi} \, |z;\lambda)(z;\lambda|\rho_\lambda^c|\sigma;\lambda)(\sigma;\lambda|. \tag{64}$$

By the diagonal representation of the unnormalized density operator we have

$$(z;\lambda|\rho_\lambda^c|z';\lambda) = P_\lambda(z^*,z';\beta)\,\delta(z-z'), \tag{65}$$

where $\delta(z-z')$ is the Dirac distribution.

Moreover, using Eqs. (60), (17), and (18), we obtain

$$(z;\lambda|A(a_\lambda^+,a)|z';\lambda) = \sum_{m,l=0}^{\infty} A_{ml}(z;\lambda|\left(a_\lambda^+\right)^m a^l|z';\lambda) =$$

$$= \sum_{m,l=0}^{\infty} A_{ml}\,(z^*+\lambda)^m\,(z')^l(z;\lambda|z';\lambda) = A(z^*+\lambda,z')(z;\lambda|z';\lambda). \tag{66}$$

In the particular case if $z'=z$, we obtain the average value of the operator A in the BDS-CS representation $|z;\lambda)$.

Then, the thermal average of an operator $A = A(a_\lambda^+,a)$ in a normal ordered form with respect to the new creation and orthodox annihilation operators can be calculated with the relation

$$\langle A\rangle = \frac{1}{\overline{n}}\int_{\mathrm{C}} \frac{d^2z}{\pi}\,\exp\left(-\frac{1}{\overline{n}}|z+\lambda|^2\right) A(z^*+\lambda,z). \tag{67}$$

This equation shows that if we have to calculate the average of the normally ordered operator $A(a_\lambda^+,a)$ then, first of all, the orthodox creation operators a^+ may be substituted by the variable z^* and the annihilation operators a may be substituted by the variable z and then we can perform the integration over the whole complex plane, according to the previous equation.

From the power series development of the operator A we can see that the ordered product of operators $\left(a_\lambda^+\right)^m a^l$ plays an important role in the calculation of averages. Taking into account that

$$\left(a_\lambda^+\right)^m a^l = \left(a^+ + \lambda\right)^m a^l = \sum_{k=0}^{m}\binom{m}{k}\lambda^{m-k}(a^+)^k a^l, \tag{68}$$

it is evident that, finally, we must calculate the thermal averages of the ordered orthodox operator product $\langle (a^+)^k\, a^l\rangle$. Using the previous ansatz, we have

$$\langle \left(a^+\right)^k a^l\rangle = \frac{1}{\overline{n}}\int_C \frac{\mathrm{d}^2z}{\pi}\,\exp\left(-\frac{1}{\overline{n}}|z+\lambda|^2\right)\,(z^*)^k\,z^l. \tag{69}$$

The complex integrals of this kind are calculated in the Appendix. Finally, we obtain

$$\langle \left(a^+\right)^k a^l\rangle = m!\,(\overline{n})^m\,(-\lambda)^{M-m}\,L_m^{M-m}\left(-\frac{1}{\overline{n}}\lambda^2\right), \tag{70}$$

where $m := \min(k,l)$, $M := \max(k,l)$, and $L_m^{M-m}(\ldots)$ is a generalized Laguerre polynomial.

This result agrees with the one obtained in Ref. [16] for the thermal states, which corresponds to the superposition of a coherent and chaotic radiation field. Also the above result coinicide with the previous results obtained by other authors [17], [18].

Using Eq. (67) we obtain

$$\langle \left(a_\lambda^+\right)^m a^l \rangle = \langle \left(a^+ + \lambda\right)^m a^l \rangle = \sum_{k=0}^{m} \binom{m}{k} \lambda^{m-k} \langle (a^+)^k a^l \rangle, \tag{71}$$

where we must take into account the ordering relation between the powers k and l, according to Eq. (69).

Following [19], let us introduce the lower generating function

$$G(\nu,\mu) := \left\langle \exp\left(\nu a^+ + \mu a\right) \right\rangle, \tag{72}$$

which, using the previous ansatz (Eq. (66)) and Eq. (38), can be written as follows:

$$G(\nu,\mu) = \exp\left[\overline{n}\left(\nu - \frac{\lambda}{\overline{n}}\right)\left(\mu - \frac{\lambda}{\overline{n}}\right) - \frac{1}{\overline{n}}\lambda^2\right]. \tag{73}$$

The practical utility of this generating function consists in the fact that the averages of the product of ordered creation and annihillation operators powers can be expressed in the following manner:

$$\langle (a^+)^k a^l \rangle = \left(\frac{\partial}{\partial \nu}\right)^k \left(\frac{\partial}{\partial \mu}\right)^l G(\nu,\mu)|_{\nu=\mu=0}. \tag{74}$$

We exemplify here some particular thermal averages for the orthodox operators

$$\langle (a^+)^k \rangle = \langle a^k \rangle = (-1)^k \lambda^k\,, \qquad \langle a^+ a \rangle = \overline{n} + \lambda^2, \tag{75}$$

$$\langle (a^+)^2 a \rangle = \langle a^+ a^2 \rangle = -2\overline{n}\lambda - \lambda^3\,, \qquad \langle (a^+)^2 a^2 \rangle = 2(\overline{n})^2 + 4\overline{n}\lambda^2 + \lambda^4. \tag{76}$$

The corresponding thermal averages for the displaced creation operator a_λ^+ are

$$\langle (a_\lambda^+)^m \rangle = 0\,, \qquad \langle a_\lambda^+ a \rangle = \overline{n}. \tag{77}$$

These averages are useful to calculate the dispersion of an observable A with respect to the thermal states, according the definition

$$\sigma_A := (\Delta A)^2 = \langle A^2 \rangle - \left(\langle A \rangle\right)^2. \tag{78}$$

The dispersions of the displaced creation and orthodox annihilation operators are

$$\sigma_{a_\lambda^+} = \langle (a_\lambda^+)^2 \rangle - \left(\langle a_\lambda^+ \rangle\right)^2 = 0\,, \qquad \sigma_a = \langle a^2 \rangle - \left(\langle a \rangle\right)^2 = 0, \tag{79}$$

while the dispersion of the number operator $N_\lambda = a_\lambda^+ a$ is

$$\sigma_{N_\lambda} = \langle N_\lambda^2 \rangle - \left(\langle N_\lambda \rangle\right)^2 = \overline{n}(\overline{n}+1). \tag{80}$$

This quantity depends only on the equilibrium temperature T (through the variable $\beta = (k_{\mathrm{B}}T)^{-1}$) and it is independent of the displacement parameter λ.

From Eqs. (11) and (12) we obtain

$$x = \frac{1}{\sqrt{2}}\sqrt{\frac{\hbar}{m\omega}}(a^{+} + a)\,, \qquad p = \mathrm{i}\frac{1}{\sqrt{2}}\sqrt{m\hbar\omega}(a^{+} - a). \tag{81}$$

The dispersion of these two operators can easy be obtained using the above calculated averages of the creation and annihilation operators:

$$\sigma_x = \frac{\hbar}{m\omega}\left(\overline{n} + \frac{1}{2}\right), \qquad \sigma_p = m\hbar\omega\left(\overline{n} + \frac{1}{2}\right). \tag{82}$$

The square of the product of these two dispersions is the uncertainty position-moment product:

$$\Delta x\,\Delta p = \hbar\left(\overline{n} + \tfrac{1}{2}\right) = \tfrac{1}{2}\hbar\coth\left(\tfrac{1}{2}\beta\hbar\omega\right) \geq \tfrac{1}{2}\hbar. \tag{83}$$

The last inequality is a consequence of the fact that the absolute temperature $T > 0$ and, as a result, the hyperbolic cotangent is always > 1. It can be seen that the uncertainty product for the BDS-Hamiltonian has the same expression as the corresponding product for the HO-1D [20].

If we calculate the uncertainty product in the BDS-CS representation, according to Eq. (66), we obtain that

$$\Delta x\,\Delta p = \tfrac{1}{2}\hbar\,, \tag{84}$$

which proves that the BDS-CS (25) are the coherent states from another perspective.

The von Neumann entropy is given by the average value of the logarithmic operator $\ln\rho_\lambda$ [12]:

$$S = -k_{\mathrm{B}}\langle\ln\rho_\lambda\rangle = k_{\mathrm{B}}\ln Z_\lambda(\beta) - k_{\mathrm{B}}\langle\ln\rho_\lambda\rangle. \tag{85}$$

Accordingly, the calculation of the entropy reduces to the problem of using the explicit form of the unnormalized density operator ρ_λ^{c}. In this sense, the diagonal coherent states representation of this operator (Eq. (63)) seems to be very useful.

Because the operator ρ_λ^{c} has the exponential structure

$$\rho_\lambda^{c} = \exp\left(-\beta H_\lambda\right), \tag{86}$$

we obtain for the entropy

$$S = -k_{\mathrm{B}}\langle\rho_\lambda\rangle = k_{\mathrm{B}}\ln Z_\lambda(\beta) + k_{\mathrm{B}}\beta\langle H_\lambda\rangle. \tag{87}$$

Using the commutation relation, we obtain the average of the BDS-Hamiltonian:

$$\langle H_\lambda\rangle = \hbar\omega\langle a_\lambda^{+}a\rangle + \frac{\hbar\omega}{2} = \hbar\omega\left(\overline{n} + \frac{1}{2}\right) = \hbar\omega\left(\frac{1}{\mathrm{e}^{\beta\hbar\omega} - 1} + \frac{1}{2}\right). \tag{88}$$

Then, the entropy follows immediately:

$$S = k_{\mathrm{B}} \left[\ln \tfrac{1}{2}\overline{n}(\overline{n}+1) + \beta\hbar\omega \left(\overline{n} + \tfrac{1}{2} \right) \right] , \tag{89}$$

which, by using the expression of the mean occupancy $\overline{n}$, becomes

$$S = k_{\mathrm{B}} \ln Z_{\lambda}(\beta) + k_{\mathrm{B}}\beta\langle H_{\lambda}\rangle = k_{\beta} \left[\left(\tfrac{1}{2}\beta\hbar\omega \right) \coth \left(\tfrac{1}{2}\beta\hbar\omega \right) - \ln 2 \sinh \left(\tfrac{1}{2}\beta\hbar\omega \right) \right] . \tag{90}$$

This expression is also identical with the corresponding entropy for the HO-1D, with the same angular frequency ω as the BDS-oscillator, which is in agreement to the previous observation.

To end this section, let us calculate the degree of coherence of l-th order [16]

$$g_{\lambda}^{(l)}(0) = \frac{\langle (a_{\lambda}^{+})^{l} a^{l} \rangle}{(\langle a_{\lambda}^{+} a \rangle)^{l}} . \tag{91}$$

Using the following expression for the generalized Laguerre polynomials [17]

$$\sum_{j=0}^{\{k,l\}} \frac{k!\, l!}{j!\,(k-j)!\,(l-j)!}\, c^{j} x^{k-j} y^{l-j} = l!\, c^{l} x^{k-l}\, L_{l}^{k-l}\Big(-\frac{xy}{c}\Big) = k!\, c^{k} x^{l-k}\, L_{k}^{l-k}\Big(-\frac{xy}{c}\Big) , \tag{92}$$

where $\{k,l\} := \min(k,l)$ and, also, Eq. (69), we can write the degree of coherence of l-th order as follows:

$$\langle (a_{\lambda}^{+})^{l} a^{l} \rangle = (-1)^{l} \lambda^{2l} \sum_{k=0}^{l} (-1)^{k} \binom{l}{k} \sum_{j=0}^{k} j! \binom{l}{j} \binom{k}{j} \left(\frac{\overline{n}}{\lambda^{2}} \right)^{j} , \tag{93}$$

$$g_{\lambda}^{(l)}(0) = (-1)^{l} \left(\frac{\lambda^{2}}{\overline{n}} \right)^{l} \sum_{k=0}^{l} (-1)^{k} \binom{l}{k} \sum_{j=0}^{k} j! \binom{l}{j} \binom{k}{j} \left(\frac{\overline{n}}{\lambda^{2}} \right)^{j} . \tag{94}$$

The first three particular values of the degree of coherence are

$$g_{\lambda}^{(1)}(0) = 1! , \qquad g_{\lambda}^{(2)}(0) = 2! , \qquad g_{\lambda}^{(3)}(0) = 3! , \tag{95}$$

and we can demonstrate that the degree of coherence of l-th order is

$$g_{\lambda}^{(l)}(0) = l! . \tag{96}$$

7 Conclusions

The coherent states of the Beckers, Debergh, and Szafraniec-Hamiltonian (BDS-CS) may be constructed in a similar way as for the usual one-dimensional harmonic oscillator. This was possible due to the use of a new measure in the Hilbert space of the BDS-Hamiltonian eigenstates, which make these functions orthonormalized and complete. The eigenfunctions, the BDS-CS and some of the averages of different physical observables concerning the BDS-Hamiltonian depend on a real parameter λ and in the limit $\lambda \to 0$ we recover the corresponding expressions for the usual one-dimensional oscillator. Because the partition function $Z_\lambda(\beta) = Z_0(\beta)$; i.e., it is independent with respect to the displacement parameter λ and it is identical with the partition function of the one-dimensional usual harmonic oscillator, all thermal averages, respectively all thermodynamical and statistical properties of a quantum system of BDS-Hamiltonian are identical with those of the HO-1D quantum system.

Beginning from other premises, namely using the considerations of the supersymmetric quantum mechanics (SUSYQM) (see, e.g., Ref. [22]), Beckers et *al.* [23] have introduced a new inner product in the Hilbert space of the Hamiltonian H_λ eigenfunctions, which can be expressed using the measure identical with our new measure (8). Several of the results obtained in Ref. [23] are complementary with some our results of the present paper.

Even if the thermal averages of the quantum canonical system of the BDS-oscillators are identical with those corresponding to the HO-1D system, the BDS-CS, being different from those for the HO-1D, may certainly furnish additional information, which makes for an interesting future examination of the BDS-Hamiltonian.

Appendix

We have to calculate the integral of the following kind:

$$I_z = \int_C \frac{\mathrm{d}^2 z}{\pi} \exp\left[-q|z|^2 - sz - tz^*\right](z^*)^k z^l, \tag{97}$$

where q is a positive quantity, s and t are real or complex numbers, while k and l are positive integer numbers. We consider the complex variable z as

$$z = r\exp(\mathrm{i}\varphi), \qquad r \in [0, +\infty], \qquad \varphi \in [0, 2\pi]. \tag{98}$$

This complex integral can be split into real and angular parts

$$I_z = \int_0^\infty \mathrm{d}r\, r^{k+l+1} \exp\left(-qr^2\right) \int_0^{2\pi} \frac{\mathrm{d}\varphi}{\pi} \exp\left[-sz - tz^* + \mathrm{i}(l-k)\varphi\right] := I_r\, I_\varphi\,. \tag{99}$$

The angular integral is a sum of two integrals:

$$I_\varphi = I_{[\cos]} + \mathrm{i} I_{[\sin]}, \tag{100}$$

where

$$I_{\left[\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\right]} := \int_0^{2\pi} \frac{\mathrm{d}\varphi}{\pi} \begin{bmatrix}\cos\\ \sin\end{bmatrix}[(l-k)\varphi] \exp\left[(s+t)r\cos\varphi + \mathrm{i}(s-t)r\sin\varphi\right], \tag{101}$$

which are two particular cases of a more general integral [21].

$$\begin{aligned}&\int_0^{2\pi} \frac{\mathrm{d}\varphi}{\pi} \begin{bmatrix}\cos\\ \sin\end{bmatrix}(p\cos\varphi + q\sin\varphi + n\varphi)\exp\left(a\cos\varphi + b\sin\varphi\right) = \\ &\quad = (-1)^\delta \frac{(A+\mathrm{i}B)^{n/2} I_n(\sqrt{C-\mathrm{i}D}) \mp (A-\mathrm{i}B)^{n/2} I_n(\sqrt{C+\mathrm{i}D})}{[(p-b)^2 + (q+a)^2]^{n/2}},\end{aligned} \tag{102}$$

where $\delta = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $I_n(x)$ are the modified Bessel functions and

$$A = a^2 - b^2 + p^2 - q^2\,, \qquad B = 2(ab + pq), \tag{103}$$

$$C = a^2 + b^2 - p^2 - q^2\,, \qquad D = 2(ap + bq), \tag{104}$$

with the conditions

$$(p - b)^2 + (q + a)^2 > 0\,, \qquad n = 0,\, 1,\, 2, \ldots \tag{105}$$

In our case,

$$p = q = 0\,, \qquad n = |l - k| = 0,\, 1,\, 2, \ldots, \qquad a = (s + t)r\,, \qquad b = \mathrm{i}(s - t)r\,, \tag{106}$$

so that

$$A = 2r^2(s^2 + t^2)\,, \qquad B = 2\mathrm{i}r^2(s^2 - t^2)\,, \qquad C = 4r^2 st\,, \qquad D = 0\,. \tag{107}$$

We obtain

$$I_{\begin{bmatrix}\cos\\ \sin\end{bmatrix}} = (-1)^\delta \left[\left(\sqrt{\frac{t}{s}} \right)^n \mp \left(\sqrt{\frac{s}{t}} \right)^n \right] I_n(2r\sqrt{st}). \tag{108}$$

The angular integral becomes

$$I_\varphi = I_{[\cos]} \pm \mathrm{i} I_{[\sin]} = 2 \left(\frac{t}{s} \right)^{(l-k)/2} I_n(2r\sqrt{st}) = 2 \left(\frac{s}{t} \right)^{(k-l)/2} I_n(2r\sqrt{st})\,, \tag{109}$$

the first sign and result correspond to the case $l > k$, while the second to the case $l < k$.

The complex integral is then

$$\begin{aligned} I_z &= 2 \left(\frac{t}{s} \right)^{(l-k)/2} \int_0^\infty \mathrm{d}r\, r^{l+k+1} \exp{(-qr^2)}\, I_n(2r\sqrt{st}) \\ &= 2 \left(\frac{s}{t} \right)^{(k-l)/2} \int_0^\infty \mathrm{d}r\, r^{l+k+1} \exp{(-qr^2)}\, I_n(2r\sqrt{st})\,. \end{aligned} \tag{110}$$

When we perform the variable change

$$x = q\sqrt{r} \tag{111}$$

and use the relation between the Bessel functions of the real and imaginary argument, i.e.,

$$I_n(x) = (-1)^n J_n(-\mathrm{i}x), \tag{112}$$

then we obtain the integrals

$$\begin{aligned} I_z &= 2(-\mathrm{i})^n \left(\frac{t}{s} \right)^{(l-k)/2} q^{(l+k+2)/2} \int_0^\infty \mathrm{d}x\, x^{l+k+1} \exp{(-x^2)}\, I_n\left(2\mathrm{i}x \sqrt{\frac{st}{q}} \right) \\ &= 2(-\mathrm{i})^n \left(\frac{s}{t} \right)^{(k-l)/2} q^{(l+k+2)/2} \int_0^\infty \mathrm{d}x\, x^{l+k+1} \exp{(-x^2)}\, I_n\left(2\mathrm{i}x \sqrt{\frac{st}{q}} \right). \end{aligned} \tag{113}$$

Now, we can use the following integral [21]:

$$\int_0^\infty \mathrm{d}x\, \mathrm{e}^{-x^2}\, x^{2n+\mu+1}\, J_\mu(2x\sqrt{z}) = \frac{1}{2}\, n!\, \mathrm{e}^{-z}\, z^{\mu/2} L_n^\mu(z), \tag{114}$$

with the conditions: $n = 0,\, 1,\, 2, \ldots$ and $n + \operatorname{Re}\mu > -1$. Here $L_n^\mu(z)$ are the generalized Laguerre's polynomials.

After the straightforward calculations, we obtain the integral (96), in compact notation.

$$I_z = \int_C \frac{\mathrm{d}^2 z}{\pi}\, \exp\left[-q|z|^2 - sz - tz^*\right](z^*)^k z^l = \tag{115}$$

$$= \frac{1}{q}\, k!\, t^{l-k} \frac{1}{q^l} \exp\left(\frac{st}{q}\right) L_k^{l-k}\left(-\frac{st}{q}\right) = \frac{1}{q}\, l!\, t^{k-l} \frac{1}{q^k} \exp\left(\frac{st}{q}\right) L_l^{k-l}\left(-\frac{st}{q}\right).$$

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