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Some Properties of Generalized Hypergeometric Thermal Coherent States

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Abstract: The generalized hypergeometric coherent states (GHCSs) have been introduced by Appl and Schiller [1]. In the present paper we have extended some considerations about GHCSs to the mixed (thermal) states and applied, particularly, to the case of pseudoharmonic oscillator (PHO). The Husimi's Q distribution function and the diagonal P - distribution function, in the GHCSs representation, have been deduced for these mixed states. The obtained distribution functions were used to calculate thermal averages and to examine some nonclassical properties of the generalized hypergeometric thermal coherent states (GHTCSs), particularly for the PHO. We have also defined and calculated the thermal analogue of the Mandel parameter and the thermal analogue of the second-order correlation function. By particularizing the parameters p and q of the hypergeometric functions, we recover the usual Barut-Girardello coherent states and their main properties for the PHO from our previous paper [9]. All calculations are performed in terms of the Meijer's G-functions [2], which are related to the hypergeometric functions. This manner provides an elegance and uniformity of the obtained results and so the GHCSs become a new field of application for these functions. Moreover, this mathematical approach can be used also for other kind of coherent states (e.g. Klauder-Perelomov, Gazeau-Klauder or nonlinear coherent states [10], [12]).

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1. Introduction

By considering the so-called generalized lowering and raising operators

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$$ {}_pU_q = \sum_{n=0}^{\infty} {}_pf^q(n)|\lambda;n >< \lambda;n+1|, \tag{1} $$

$$ {}_pU_q^+ = \sum_{n=0}^{\infty} {}_pf^q(n)|\lambda;n+1 >< \lambda;n| \tag{2} $$

where $|\lambda;n>$ are the vectors of an orthonormal basis (usually, the Fock basis indexed by a real parameter λ which can play the role of the degeneracy parameter), Appl and Schiller have defined the generalized hypergeometric coherent states (GHCSs) as the eigenvalues of the lowering operator [1]:

$$ {}_pU_q|p;q;\lambda;z >= z|p;q;\lambda;z > . \tag{3} $$

The numbers p and q are natural and the positive functions ${}_pf^q(n)$ were defined as follows:

$$ {}_pf^q(n) = \sqrt{(n+1)\frac{(b_1+n)(b_2+n)\dots(b_q+n)}{(a_1+n)(a_2+n)\dots(a_p+n)}}. \tag{4} $$

Consequently, the expansion of the GHCSs in the Fock basis is [1]:

$$ |p;q;\lambda;z >= \left[{}_pN_q(|z|^2;\lambda)\right]^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{{}_p\rho_q(n;\lambda)}}|\lambda;n> \tag{5} $$

where the strictly positive parameter functions of the discrete variable n are defined as:

$$ {}_p\rho_q(n;\lambda) = \Gamma(n+1)\frac{\prod_{j=1}^{q}(b_j)_n}{\prod_{i=1}^{p}(a_i)_n} \tag{6} $$

and where $(x)_n = \Gamma(x+n)/\Gamma(x)$ is the Pochhammer's symbol [2].

The appellation "generalized hypergeometric coherent states" becomes from the normalization function which is given by generalized hypergeometric functions:

$$ \begin{aligned} {}_pN_q(|z|^2;\lambda) &= {}_pF_q(a_1,\, a_2, \dots, a_p; b_1,\, b_2, \dots, b_q; |z|^2) = \\ &= \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n\dots(a_p)_n}{(b_1)_n(b_2)_n\dots(b_q)_n}\frac{(|z|^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{{}_p\rho_q(n;\lambda)}(|z|^2)^n, \end{aligned} \tag{7} $$

where, generally, we consider that $a_i = a_i(\lambda)$ and $b_j = b_j(\lambda)$ are the complex functions.

In the cited paper [1] were examined the main properties of the GHCSs, including the resolution of unity. Let us here we express the weight function of the integration measure through the Meijer's G-functions (whose definition and main properties can be found, e.g. in [2]):

$$d\mu(p;q;\lambda;z) = \frac{d^2 z}{\pi}\, {}_p w_q(|z|^2;\lambda) = \tag{8}$$
$$= \frac{d^2 z}{\pi}\, G^{1,p}_{p,q+1}\left(-|z|^2 \left| \begin{matrix} \{1-a_p\}\,;/ \\ 0\,;\{1-b_q\} \end{matrix} \right.\right) G^{q+1,0}_{p,q+1}\left(|z|^2 \left| \begin{matrix} /;\,\{a_p-1\} \\ 0,\,\{b_q-1\}\,;/ \end{matrix} \right.\right)$$

where, in order to simplify the formulae writing, we have used the following notation: $\{c_l\} \equiv c_1,\, c_2, \ldots, c_l$.

Below we have also used the connection between the generalized hypergeometric functions and the Meijer's G-functions [2]:

$${}_pF_q(\{a_p\};\{b_q\};x) = \frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{i=1}^{p}\Gamma(a_i)}\, G^{1,p}_{p,q+1}\left(-x \left| \begin{matrix} \{1-a_p\}\,;/ \\ 0\,;\{1-b_q\} \end{matrix} \right.\right). \tag{9}$$

2. GHCSs for Thermal States

In [1] were examined some properties of the GHCSs for the case of pure quantum states. One of the aims of present paper is to extend this examination also to the case of the mixed quantum states. As a typical example of mixed states we consider the thermal states described by the normalized canonical density operator:

$$\rho_\lambda = \frac{1}{Z_\lambda}\sum_{n=0}^{\infty} e^{-\beta E_{n,\lambda}} |\lambda; n><\lambda; n| \tag{10}$$

where $E_{n,\lambda}$ are the eigenvalues of the Hamiltonian operator of the examined quantum system and the normalization constant $Z_\lambda = Z_\lambda(\beta)$ is the partition function.

In the present paper we will limiting only to the case of quantum systems with linear energy spectra with respect to the energy quantum number n and we will to describe the corresponding method for examine some properties of generalized hypergeometric thermal coherent states (GHTCSs). For systems with more complicated energy spectra it must elaborate specifical methods (e.g. for the Morse oscillator, see [3]).

Let us we assume that the energy of linear spectra (where $n = 0, 1, \ldots, \infty$ and ω is the angular frequency) is

$$E_{n,\lambda} = E_{0,\lambda} + n\hbar\omega \tag{11}$$

and then the partition function becomes:

$$Z_\lambda = \sum_{n=0}^{\infty} e^{-\beta E_{n,\lambda}} = e^{-\beta E_{0,\lambda}} (<n> + 1) \tag{12}$$

where we have used the expression of the Bose-Einstein distribution function:

$$< n >= \frac{1}{e^{\beta\hbar\omega} - 1}. \tag{13}$$

The first characteristic of the GHTCSs we want to examine is the thermal Husimi's distribution function:

$$Q_{|p;q;\lambda;z>}(\rho_\lambda) \equiv < p; q; \lambda; z|\rho_\lambda|p; q; \lambda; z >= \frac{1}{Z_\lambda}\frac{1}{{}_pF_q(\{a_p\}; \{b_q\}; |z|^2)}\sum_{n=0}^{\infty} e^{-\beta\, E_{n,\lambda}}\frac{(|z|^2)^n}{{}_p\rho_q(n; \lambda)}. \tag{14}$$

For the above examined linear spectra this expression becomes:

$$Q_{|p;q;\lambda;z>}(\rho_\lambda) = \frac{1}{< n > +1}\frac{{}_pF_q(\{a_p\}; \{b_q\}; \frac{<n>}{<n>+1}|z|^2)}{{}_pF_q(\{a_p\}; \{b_q\}; |z|^2)} = \tag{15}$$

$$= \frac{1}{< n > +1}\frac{G^{1,p}_{p,q+1}\left(-\frac{<n>}{<n>+1}|z|^2\left|\begin{array}{l}\{1 - a_p\}\,;\,/\\ 0\,;\{1 - b_q\}\end{array}\right.\right)}{G^{1,p}_{p,q+1}\left(-|z|^2\left|\begin{array}{l}\{1 - a_p\}\,;\,/\\ 0\,;\{1 - b_q\}\end{array}\right.\right)}.$$

Using the integration measure expression and the properties of the integral of Meijer's G-functions products [2], it is not difficult to prove that the thermal Husimi's distribution function is positive, normalized to unity with the measure $d\mu(p; q; \lambda; z)$ and provides a two-dimensional probability distribution over the complex $z = |z| \exp{(\mathrm{i}\varphi)}$ plane:

$$\int d\mu(p; q; \lambda; z)Q_{|p;q;\lambda;z>}(\rho_\lambda) = 1. \tag{16}$$

After these considerations let us we perform the diagonal expansion of the density operator ρ_λ over the projector of GHTCSs:

$$\rho_\lambda = \int d\mu(p; q; \lambda; z)|p; q; \lambda >\ {}_pP_q(|z|^2; \lambda) < p; q; \lambda; z| \tag{17}$$

where the function ${}_pP_q(|z|^2; \lambda)$ is called P-function or P-distribution function, even if is in fact a quasi-probability distribution, because of their negative values on certain domains.

By inserting the expressions for the GHTCSs and the integration measure, performing the angular integration and using the notation $x = |z|^2$, we lead to the following equation:

$$\rho_\lambda = \frac{\prod_{i=1}^{p}\Gamma(a_i)}{\prod_{j=1}^{q}\Gamma(b_j)}\sum_{n=0}^{\infty}\frac{|\lambda; n >< \lambda; n|}{{}_p\rho_q(n; \lambda)}\int_0^{R^2} dx\, x^n\, G^{q+1,0}_{p,q+1}\left(x\left|\begin{array}{l}/;\ \{a_p - 1\}\\ 0,\ \{b_q - 1\}\,;\,/\end{array}\right.\right){}_pP_q(x; \lambda) \tag{18}$$

where R is the convergence radius of the GHTCSs [5].

If we perform the function change

$$ {}_pP_q(x;\lambda) = \left[G^{q+1,0}_{p,q+1}\left(x \left| \begin{matrix} /;\ \{a_p-1\} \\ 0,\ \{b_q-1\}\ ;\ / \end{matrix} \right. \right) \right]^{-1} {}_ph_q(x;\lambda) \tag{19} $$

and consider the expression of strictly positive parameters functions ${}_p\rho_q(n;\lambda)$ and the energy $E_{n,\lambda}$, in order to obtain the expression (10) for the normalized canonical density operator ρ_λ, we must solve the following equation where ${}_ph_q(x;\lambda)$ is, for the moment, an unknown function which must be determined:

$$ \int_0^{R^2} dx\, x^n\, {}_ph_q(x;\lambda) = \frac{1}{Z_\lambda} e^{-\beta\, E_{0,\lambda}} \frac{1}{\left(e^{\beta\hbar\omega}\right)^n} \Gamma(n+1) \frac{\prod_{j=1}^q \Gamma(b_j+n)}{\prod_{i=1}^p \Gamma(a_i+n)}. \tag{20} $$

The above problem is just the moment power problem (if R is a finite quantity we have the Haussdorff moment problem, while if R is infinite we have the Stieltjes moment problem [4]) and, in order to solve it, we extend the real values n to the complex ones $s = n+1$ [5] and so we lead to the definition of the Meijer's G-functions [2]. In this stage the problem can be solved.

Finally, for the P-function we obtain:

$$ {}_pP_q(|z|^2;\lambda) = \frac{1}{<n>} \frac{G^{q+1,0}_{p,q+1}\left(\frac{<n>+1}{<n>}|z|^2 \left| \begin{matrix} /;\ \{a_p-1\} \\ 0,\ \{b_q-1\}\ ;\ / \end{matrix} \right. \right)}{G^{q+1,0}_{p,q+1}\left(|z|^2 \left| \begin{matrix} /;\ \{a_p-1\} \\ 0,\ \{b_q-1\}\ ;\ / \end{matrix} \right. \right)} \tag{21} $$

Using the properties of Meijer's G-functions [2] it is not difficult to prove that the P-function is normalized to unity:

$$ \int d\mu(p;q;\lambda;z)_p P_q(|z|^2;\lambda) = 1. \tag{22} $$

By particularizing the parameters p and q of the hypergeometric functions, it must examine the positivity of the weight function of the integration measure ${}_pw_q(|z|^2;\lambda)$ (see, also [1]) and of the Husimi's function $Q_{|p;q;\lambda;z>}(\rho_\lambda)$. Implicitly from their definition, the Husimi's function $Q_{|p;q;\lambda;z>}(\rho_\lambda)$ is always positive, while the P-function can bring also negative and singular values (for non-classical fields or states). For these reasons, the P-function is sometimes called the quasi-distribution function. If the P-function becomes negative for a certain state, then this state has a non-classical character [6].

One of the practical utility of the P-function of a density operator (which can be measured in experiments) consists in its role in the calculation of thermal averages of an observable A which characterize the quantum system:

$$< A >_{p;q;\lambda}= Tr(\rho_\lambda A) = \int d\mu(p;q;\lambda;z)\,{}_pP_q(|z|^2;\lambda) < p;q;\lambda;z|A|p;q;\lambda;z > . \tag{23}$$

On the other hand, if the P-function has non-classical character (e.g. for a pair-coherent state [7]), then the state is entangled. This will be a good test for the inseparability of the quantum states, particularly for the coherent states.

In many cases the calculations in the GHTCSs representation are much simpler that in other representations (e.g. in the coordinate or in the momentum representations), which constitute a good reason for using the GHTCSs formalism.

An important class of observables is represented by the diagonal operators in the Fock Basis $|\lambda;n>$. As an example we examine the thermal averages of integer powers of the number operator $< N^s >$, where $s = 1, 2 \ldots$ In this reason we adopt an original ansatz. If we calculate the average of the operator $e^{\varepsilon N}$ in a pure GHTCS $|p;q;\lambda;z>$, where ε is a small positive parameter, i.e.

$$< e^{\varepsilon N} >_{|p;q;\lambda;z>}= \frac{{}_pF_q(\{a_p\};\{b_q\};e^{\varepsilon}|z|^2)}{{}_pF_q(\{a_p\};\{b_q\};|z|^2)} = \frac{G^{1,p}_{p,q+1}\left(-e^{\varepsilon}|z|^2 \left| \begin{array}{l} \{1-a_p\}\,;\,/ \\ 0\,;\{1-b_q\} \end{array}\right.\right)}{G^{1,p}_{p,q+1}\left(-|z|^2 \left| \begin{array}{l} \{1-a_p\}\,;\,/ \\ 0\,;\{1-b_q\} \end{array}\right.\right)}, \tag{24}$$

we observe that

$$< N^s >_{|p;q;\lambda;z>}= \lim_{\varepsilon\to 0}\left(\frac{\partial}{\partial\varepsilon}\right)^s < e^{\varepsilon N} >_{|p;q;\lambda;z>} . \tag{25}$$

Then the corresponding thermal averages are

$$< N^s >= \int d\mu(p;q;\lambda;z)\,{}_pP_q(|z|^2;\lambda) < N^s >_{|p;q;\lambda;z>}= \frac{1}{< n >}\lim_{\varepsilon\to 0}\left(\frac{\partial}{\partial\varepsilon}\right)^s \frac{1}{\frac{<n>+1}{<n>} - e^{\varepsilon}}, \tag{26}$$

where we have used the integral properties of the Meijer's G-functions and also the particular expression for the function $G^{11}_{11}(x|\ldots)$ [2].

We can observe that these thermal averages are independent of the parameters p, q and λ, which was to be expected, due to the fact that the thermal averages are independent on the representation.

The first two power thermal averages can be expressed through the Bose-Einstein distribution function (13):

$$< N >=< n >, \qquad\qquad < N^2 >=< n > (2 < n > +1). \tag{27}$$

With these thermal averages we can define and calculate the thermal second-order correlation function $g^{(2)}$ and the thermal Mandel parameter $Q(< n >)$ [3], which are defined as the thermal analogues of the corresponding quantities for the pure GHCS $|p; q; \lambda; z >$ [6]:

$$g^{(2)} = \frac{< N^2 > - < N >}{(< N >)^2} = 2, \qquad Q(< n >) =< N > \left(g^{(2)} - 1\right) =< n > . \quad (28)$$

So, all GHTCSs, independently of the values of integer parameters p, q and λ, have the same (constant) thermal second-order correlation function $g^{(2)}$ and the same thermal Mandel parameter $Q(< n >)$ (which are dependent of the equilibrium temperature T, through the quantity $\beta = (k_B T)^{-1}$).

3. Particularization for the Pseudoharmonic Oscillator

In the following, we particularize the GHTCSs formalism for the case of the pseudo-harmonic oscillator (PHO), whose effective potential is [8]

$$V_J(r) = \frac{m_{red}\omega^2}{8} r_0^2 \left(\frac{r}{r_0} - \frac{r_0}{r}\right)^2 + \frac{\hbar^2}{2m_{red}} J(J+1) \frac{1}{r^2} \quad (29)$$

where m_{red} is the reduced mass of the quantum system (e.g. the diatomic molecule), r_0 is the equilibrium distance between the diatomic molecule nuclei and $J = 0, 1, 2, \ldots$ is the rotational quantum number, while n will be the vibrational quantum number.

The importance of the PHO consists in the fact that this potential also admits the exact analytical solution of the rotational-vibrational Schrödinger equation, being in a certain sense an intermediate potential between the three-dimensional harmonic oscillator potential (HO-3D)(an ideal potential) and other much anharmonic oscillator potentials (the more realistic potentials, e.g. Pöschl-Teller, or Morse [3]).

Earlier we have showed [9] that the dynamical group associated with the PHO is $SU(1,1)$ quantum group, whose lowering operator is

$$K_- = \sum_{n=0}^{\infty} \sqrt{(n+1)(2k+n)} |k; n >< k; n+1| \quad (30)$$

where k is the Bargmann index which labels the irreducible representations of this group.

By comparing this expression with that of generalized hypergeometric operator ${}_pU_q$ and also with the positive functions ${}_pf^q(n)$ [1], we observe that the above expression is a particular case, if we take $p = 0$, $q = 1$, $b_1 = 2k$ and $\lambda = k$. So, the GHCSs denoted as $|0;\, 1,\, k; z >= |k; z >$, defined as the eigenstates of the lowering operator K_-, i.e.

$$K_-|k; z >= z|k; z >, \quad (31)$$

are called the Barut-Girardello coherent states for the PHO and their expansion over the Fock vector basis $|k; n >$ is [9]

$$|k; z> = \sqrt{\frac{|z|^{2k-1}}{I_{2k-1}(2|z|)}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!\Gamma(2k+n)}} |k; n> . \tag{32}$$

In the following we look over all the quantities which characterize the CSs of the PHO, previously obtained in [9], but now regarded only as the particular case of the GHCSs, with the above mentioned values of the parameters p, q, b_1 and λ.

The weight function of the integration measure finally is expressed as a product of two Bessel functions of the first and second kind:

$$_0w_1(|z|^2; \lambda) = G^{1,0}_{0,2}(-|z|^2|0;\, 1-2k)\, G^{2,0}_{0,2}(|z|^2|0,\, 2k-1) = 2I_{2k-1}(2|z|)\, K_{2k-1}(2|z|). \tag{33}$$

On the other hand, the PHO energy eigenvalues are

$$E_{n,k} = \hbar\omega\, k - \frac{m_{red}\omega^2}{2\hbar} r_0^2 + \hbar\omega\, n \equiv E_{0,k} + \hbar\omega\, n. \tag{34}$$

while the connection between the Bargmann index k and the rotational quantum number J is [9]

$$k = k(J) \equiv \frac{1}{2} + \frac{1}{2}\left[\left(J + \frac{1}{2}\right)^2 + \left(\frac{m_{red}\omega}{2\hbar} r_0^2\right)^2\right]^{\frac{1}{2}} . \tag{35}$$

Accordingly to this, the partition function for a fixed quantum number J becomes

$$Z_k \equiv Z_J = \sum_{n=0}^{\infty} e^{-\beta E_{n,k}} = e^{-\beta E_{0,k}} \left(< n > +1\right). \tag{36}$$

Similarly, the total partition function (by considering also the degeneration of the rotational states) is

$$Z = \sum_{J=0}^{\infty} (2J+1) Z_J = e^{\beta \frac{m_{red}\omega^2}{4} r_0^2} \left(< n > +1\right) \sum_{J=0}^{\infty} (2J+1) \left(\frac{< n >}{< n > +1}\right)^{k(J)} . \tag{37}$$

The total partition function is a quantity of exceptional information importance because it enables the calculations of all statistical properties of a PHO quantum canonical gas (for details, see [9]).

For the thermal Husimi's distribution function for the PHO it follows:

$$\begin{aligned} Q_{|0;1;k;z>}(\rho_k) &= \frac{1}{< n > +1} \frac{G^{1,0}_{0,2}\left(-\frac{<n>}{<n>+1}|z|^2 \middle| 0;\, 1-2k\right)}{G^{1,0}_{0,2}\left(-|z|^2 \middle| 0;\, 1-2k\right)} = \\ &= \frac{1}{< n > +1} \left(\frac{< n > +1}{< n >}\right)^{k-\frac{1}{2}} \frac{I_{2k-1}\left(2|z|\sqrt{\frac{<n>}{<n>+1}}\right)}{I_{2k-1}(2|z|)}. \end{aligned} \tag{38}$$

On the other hand, for the P-function of the PHO we obtain:

$$
{}_0P_1(|z|^2;k) = \frac{1}{<n>} \frac{G_{0,2}^{2,0}\left(\frac{<n>+1}{<n>}|z|^2 \middle| 0,\, 2k-1\right)}{G_{0,2}^{2,0}\left(|z|^2 \middle| 0,\, 2k-1\right)} = \qquad (39)
$$

$$
= \frac{1}{<n>} \left(\frac{<n>+1}{<n>}\right)^{k-\frac{1}{2}} \frac{K_{2k-1}\left(2|z|\sqrt{\frac{<n>+1}{<n>}}\right)}{K_{2k-1}(2|z|)}.
$$

In both equations, in order to express the particular values of the Meijer's G-functions, we have used the book of Mathai and Saxena [2].

Because, as we have seen, the thermal averages of the integer powers of the number operator are independent of the parameters p and q of the hypergeometric functions, the indicated values in the previous section are identical both for the CSs of the PHO and for the GHCSs with arbitrarily parameters p and q . This property also pass to all statistical averages of the diagonal operators in the Fock basis.

4. Concluding Remarks

In the paper we have shown that the formalism of the GHCSs, previously introduced by Appl and Schiller for the pure quantum (coherent) states [1], can be extended also to the mixed (thermal) quantum states and in this manner it can be connected with more practical, quantum statistical, problems.

We have showed that by particularizing the parameters of the hypergeometic functions ($p = 0$ and $q = 1$) and also for the particularly value of parameter λ (i.e. $\lambda = k$ and $b_1 = 2k$) we recover the Barut-Girardello coherent states (BG-CSs) for the pseudoharmonic oscillator (PHO), deduced in a previous paper [9], with all statistical properties evinced therein.

It is important to point out that we have performed all calculations in terms of the Meijer's G-functions [2], which are more generally functions and whose particular cases are also the hypergeometric functions. This manner provide an elegance and uniformity in the use of the GHTCSs formalism and, implicitly, indicates a new field of applications of these functions in theoretical physics.

Moreover, the mathematical approach of the GHTCSs can be used also for other kind of thermal coherent states, e.g. Klauder-Perelomov, Gazeau-Klauder (for PHO these CSs were deduced in [10]) or for nonlinear CSs [11]. This approach can be also applied in the theory of quantum information [12].

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