Pair-coherent states of the pseudoharmonic oscillator

D Popov¹ and V Sajfert²

¹ University 'Politehnica' of Timisoara, B-dul V Parvan No. 2, 300223 Timisoara, Romania
 ² Technical Faculty 'M Pupin' Zrenjanin, University of Novi Sad, Djure Djakovica bb, 23000 Zrenjanin, Serbia

E-mail: dusan_popov@yahoo.co.uk and sajfertv@vektor.net

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Abstract

In the paper, we have constructed and examined some properties of pair-coherent states of Barut–Girardello kind for two noninteracting subsystems of pseudoharmonic oscillators. The pseudoharmonic oscillator obeys SU(1,1) group symmetry, which was extensively used to study many problems in quantum optics and quantum information theory.

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1. Introduction

As is well known, groups involving simple Lie algebras (and among them SU(1,1)) have been used to study many problems in quantum optics [1]. Besides the one-dimensional harmonic oscillator (HO-1D), an oscillator with many possibilities of applications in quantum optics is the pseudoharmonic oscillator (PHO) [2–4], whose effective potential is

$$V_J(r) = \frac{m_{\rm r}\omega^2}{8}r_0^2 \left(\frac{r}{r_0} - \frac{r_0}{r}\right)^2 + \frac{\hbar^2}{2m_{\rm r}}J(J+1)\frac{1}{r^2}, \quad (1)$$

where m_r is the reduced mass of the oscillating system (e.g. the nuclei of a diatomic molecule), ω the angular frequency, r_0 the equilibrium bond length or the equilibrium distance and J the rotational quantum number. Like HO-1D, the PHO also admits the exact analytical solution of the Schrödinger equation (see e.g. [2, 4, 5] and references therein). This potential obeys SU(1,1) group symmetry, i.e. its raising and lowering operators satisfy the commutation relation of this group. Consequently, in a series of previous papers we have constructed and studied some propertires of the corresponding different kinds of associated coherent states (CSs) of PHO, i.e. Barut–Girardello [3], Gazeau–Klauder [6] and Klauder-Perelomov [7]. The aim of this paper is to construct and examine some properties of pair-coherent states (pair-CSs) for two noninteracting subsystems (a and b) of PHOs and to indicate their possible applications in the theory of quantum information.

2. Pair-CSs

The PHO obeys SU(1,1) group symmetry, with the generators $(K_1, K_2 \text{ and } K_3)$ and the raising and lowering generators $K_{\pm} = K_1 \pm iK_2$ [3]. Here, we consider only the representations for which the operator K_3 is diagonal and has a discrete spectrum. The corresponding Hilbert space is spanned by the complete orthonormal basis $|n; k\rangle$ (where the real number *k* is the Bargmann index and n = 0, 1, 2, ...) and the ground state $|0; k\rangle$ is given by the condition $K_-|0; k\rangle = 0$, all states being obtained by the action of the raising operator K_+ (where $\Gamma(x)$ is Euler's Gamma function):

$$|n;k\rangle = \left[\frac{\Gamma(2k)}{\Gamma(n+1)\Gamma(2k+n)}\right]^{1/2} (K_{+})^{n} |0;k\rangle.$$
(2)

The energy eigenvalues of the PHO are well known: $E_{nJ} = \hbar \omega (n+k) - m \omega^2 r_0^2 / 4$ [2]; up an additional constant, the energy spectrum of the PHO has the same structure as the energy spectrum of the HO-1D, both the spectra being linear in the vibrational quantum number *n*. We have used this fact to construct the different kinds of CSs (Barut–Girardello and Klauder–Perelomov [6]), but in the present paper, we focus our attention on the Barut–Girardello pair-CSs in the case of a bipartite composite quantum system and on their possible applications in quantum optics and quantum information theory. Let us introduce a two-mode Fock basis $|n, m; k\rangle = |n; k\rangle \otimes |m; k\rangle$ that characterizes a bipartite system of PHOs of the same reduced mass m_r and angular frequency ω (e.g. a quantum gas of PHOs or a two-mode field). A realization of SU(1,1) group algebra in terms of two independent modes of the field, denoted by a and b, is given by the following operators:

$$\begin{split} K_{+}^{(ab)} &\equiv K_{+}^{(a)} K_{+}^{(b)}, \qquad K_{-}^{(ab)} &\equiv K_{-}^{(a)} K_{-}^{(b)}, \\ K_{3}^{(ab)} &\equiv \frac{1}{2} \left(K_{3}^{(a)} + K_{3}^{(b)} \right). \end{split}$$
(3)

The action of generators of mode *a* (similarly for mode *b*) on $|n, m; k\rangle$ is

$$K_{+}^{(a)}|n,m;k\rangle = \sqrt{(n+1)(n+2k)}|n+1,m;k\rangle, \qquad (4)$$

$$K_{-}^{(a)}|n,m;k\rangle = \sqrt{n(n+2k-1)}|n-1,m;k\rangle,$$
(5)

$$K_{3}^{(a)}|n,m;k\rangle = (n+k)|n,m;k\rangle.$$
 (6)

The generators of mode *a* are independent from those of mode *b*, so that they commute: $[K_{(...)}^{(a)}, K_{(...)}^{(b)}] = 0$, where the subscript (\cdots) is \pm or 3. Consequently, the above-defined two-mode operators also satisfy SU(1,1) group algebra, i.e. fulfil the following commutation relations:

$$\begin{bmatrix} K_3^{(ab)}, \ K_{\pm}^{(ab)} \end{bmatrix} = \pm K_{\pm}^{(ab)},$$

$$\begin{bmatrix} K_{-}^{(ab)}, \ K_{\pm}^{(ab)} \end{bmatrix} = 2K_3^{(ab)}.$$
(7)

For two independent boson annihilation operators, a pair-CS $|z; q\rangle$ was firstly defined by Agarwal [8] as an eigenstate of both the pair annihilation and the number difference operators. But here, because the number operator is defined with connection to the generator $K_3^{(...)}$, i.e.

$$N^{(a)} = K_3^{(a)} - k$$
, $N^{(b)} = K_3^{(b)} - k$, (8)

the pair-CSs for the PHO must be defined slightly differently, i.e. as an eigenstate of both raising operators $K_{-}^{(ab)} \equiv K_{-}^{(a)} K_{-}^{(b)}$ and the difference operator of $K_{3}^{(...)}$, i.e. $\Delta N^{(ab)} \equiv K_{3}^{(a)} - K_{3}^{(b)}$

$$K_{-}^{(ab)}|z;q\rangle = z|z;q\rangle, \tag{9}$$

$$\Delta N^{(ab)}|z;q\rangle = q|z;q\rangle,\tag{10}$$

where z is the complex variable and q > 0 is the degeneracy parameter whose significance will be identified below. Due to the presence of the Bargmann index k, as an additional parameter, these states can also be regarded as deformed pair-CSs. The action of the operators $K_{-}^{(ab)}$ and $\Delta N^{(ab)}$ on the vectors $|n, m; k\rangle$ is

$$K_{-}^{(ab)}|n,m;k\rangle = \sqrt{n(n+2k-1)} \\ \times \sqrt{m(m+2k-1)}|n-1,m-1;k\rangle, \quad (11)$$

$$\Delta N^{(ab)}|n,m;k\rangle = (n-m)|n,m;k\rangle.$$
(12)

So, the two-mode Fock basis $|n, m; k\rangle$ is organized in a correlation manner, i.e. the boson numbers of the two modes differ by the eigenvalue of the operator $\Delta N^{(ab)}$. Assuming that the difference between the boson numbers in both the modes is constant, without loss of generality, we can take this difference, say q, to be positive: q = 0, 1, 2, ..., i.e. q = m - n. In this case, the difference in the boson numbers

in the two modes is conserved and the state has the symmetry properties of the SU(1,1) group.

The expansion of $|z; q\rangle$ in the two-mode basis $|n, n + q; k\rangle$ is

$$|z;q\rangle = \left[{}_{0}F_{3}\left(;2k,q+1,q+2k;|z|^{2}\right)\right]^{-1/2} \\ \times \sum_{n=0}^{\infty} \frac{z^{n}}{\sqrt{\rho(n;k;q)}} |n,n+q;k\rangle,$$
(13)

where the positive quantities (called the structure constants of the pair-CSs) are

$$\rho(n;k;q) = \Gamma(n+1) \\ \times \frac{\Gamma(2k+n)}{\Gamma(2k)} \frac{\Gamma(q+1+n)}{\Gamma(q+1)} \frac{\Gamma(q+2k+n)}{\Gamma(q+2k)}$$
(14)

and where we have used the hypergeometric functions ${}_{n}F_{m}(a_{1}, \ldots, a_{n}; b_{1}, \ldots, b_{m}; x)$ [9].

In order to see that the above-obtained vectors $|z; q\rangle$ are really the coherent states, we must verify if they accomplish Klauder's minimal prescriptions [10] regarding a coherent state, i.e. a CS must (a) fulfil the identity resolution with a positive weight function; (b) be a continuous function in the complex *z*-variable, i.e. the map $z \in C \rightarrow |z\rangle \in L^2(\mathbb{R})$ must be continuous; (c) be normalizable but nonorthogonal; (d) be temporally stable; and (e) fulfil the action identity.

By taking the complex variable $z = r \exp(i\varphi)$, the resolution of identity operator

$$\int d\mu(z;q)|z;q\rangle\langle z;q| = 1$$
(15)

requires that the integration measure $d\mu(z; q)$ have the following structure:

$$d\mu(z;q) = \frac{d\varphi}{2\pi} d(r^2) \frac{{}_{0}F_3(;2k,q+1,q+2k;r^2)}{\Gamma(2k)\,\Gamma(q+1)\Gamma(q+2k)} \times G_{04}^{40}\left(r^2|0,2k-1,q,q+2k-1\right).$$
(16)

Because both functions ${}_{0}F_{3}(; ...; r^{2})$ and $G_{04}^{40}(r^{2}|...)$ cannot be expressed through elementary functions, the obtained pair-CSs $|z; q\rangle$ belong to the category of hypergeometrical coherent states [11]. For k > 1/2 both functions involved in the expression of $d\mu(z; q)$ are positive, which assures that the integration measure will also be positive. The continuity of the functions $|z; q\rangle$ in variable z is assured since for $z' \rightarrow z$ it is easy to show that we have $|z'; q\rangle \rightarrow$ $|z; q\rangle$. By calculating the overlap (or scalar product) of two pair-CSs, we can see that the states $|z; q\rangle$ are normalizable but nonorthogonal.

In order to examine the temporal stability, we consider the whole Hamiltonian of two noninteracting parts, *a* and *b*, e.g. two laser beams that are propagating independently of each other, corresponding to the two modes of the two-mode coherently correlated states: $H^{(ab)} = H^{(a)} + H^{(b)}$ [12], with the whole energy $E_n + E_{n+q} = \hbar\omega(2n+q+2k) - \frac{m\omega^2}{2}r_0^2 \equiv E_0^{(q)} + 2\hbar\omega n$.

The time-dependent pair-CSs can be defined as a result of applying the evolution operator associated with

 $H^{(ab)}$, i.e. $U(t; 0) \equiv \exp(-i/\hbar H^{(ab)} t)$, on the usual pair-CSs considered as defined for t = 0, i.e. $|z; q\rangle \equiv |z(t = 0); q\rangle$:

$$|z(t);q\rangle = U(t;0) |z;q\rangle \equiv \exp\left(-\mathrm{i} E_0^{(q)} t\right) |z\mathrm{e}^{-2\mathrm{i}\omega t};q\rangle, \quad (17)$$

which shows that, apart from an exponential oscillatory term, a pair-CS remains coherent with the complex argument depending exponentially on the time. In other words, the time-dependent pair-CS oscillates around the corresponding time-independent pair-CS.

After these calculations, we can conclude that our obtained ket vectors $|z; q\rangle$ are really coherent states.

3. Properties of the pair-CSs

For an observable A that characterizes the bipartite quantum system, the expression of averages or expectation values in the pair-CSs representation is

$$\langle z; q | A | z; q \rangle = \frac{1}{F} \sum_{n,n'=0}^{\infty} \frac{z^{*n} z^{n'}}{\sqrt{\rho(n;q;k)\rho(n';q;k)}} \\ \times \langle n, n+q; k | A | n', n'+q; k \rangle.$$
 (18)

Here, we have used the notation $F \equiv {}_{0}F_{3}(; 2k, q+1, q+2k; |z|^{2})$ and in the next, with F', F'', \ldots , we will denote the first, second, ..., derivatives with respect to the argument $|z|^{2}$.

The aim of this section is to examine the statistical properties of the pair-CSs, and consequently we need to calculate the mean values only of observables that are integer powers of the number operators:

$$N^{(a)} = K_3^{(a)} - k , \qquad N^{(b)} = K_3^{(b)} - k, \qquad (19)$$

$$N^{(ab)} = N^{(a)} + N^{(b)} = 2(K_3^{(ab)} - k),$$
(20)

whose actions on the bipartite basis vectors are

$$N^{(a)}|n, n+q; k\rangle = n|n, n+q; k\rangle, \tag{21}$$

$$N^{(b)}|n, n+q; k\rangle = (n+q)|n, n+q; k\rangle,$$
(22)

$$N^{(ab)}|n, n+q; k\rangle = (2n+q)|n, n+q; k\rangle.$$
(23)

The mean values of the first two integer powers, i.e.

$$\langle N^{(ab)} \rangle_{z;q} = q + 2|z|^2 \frac{1}{F} F',$$
 (24)

$$\left\langle [N^{(ab)}]^2 \right\rangle_{z;q} = 5q^2 + 4(2q+1)|z|^2 \frac{1}{F}F' + 4|z|^4 \frac{1}{F}F'', \quad (25)$$

are useful to calculate the Mandel parameter, defined as [3] (which indicates the behaviour of pair-CSs $|z; q\rangle$)

$$Q_{z;q} \equiv \langle N^{(ab)} \rangle_{z;q} \times \left[\frac{\left\langle \left[N^{(ab)} \right]^2 \right\rangle_{z;q} - \langle N^{(ab)} \rangle_{z;q}}{\left(\langle N^{(ab)} \rangle_{z;q} \right)^2} - 1 \right].$$
(26)

For the pair-CSs of the PHO, this expression becomes

$$Q_{z;q} = (4q-1)\frac{q+|z|^2F'}{q+2|z|^2F'} + 4|z|^4\frac{FF''-(F')^2}{F(q+2|z|^2F')}.$$
 (27)

Table 1. Behaviour of pair-CSs.pair-CSs Large
$$|z|, |z| \to \infty$$
 $q \neq 0$ $Q_{z;q} = \frac{(4q-1)(q+\sqrt{|z|})}{q+2\sqrt{|z|}}$ $Q_{z;q} = \frac{(4q-1)(q+|z|^2)}{q+2|z|^2}$ $q \neq 0$ $Q_{z;q} = 1$ supra-Poissonian $Q_{z;o} = -0.5$ $q = 0$ $Q_{z;o} = -0.5$ $Q_{z;o} = -0.5$ sub -Poissonian $Q_{z;o} = -0.5$

It is interesting to examine the asymptotic behaviour of these expressions for small and large values of |z| to determine the behaviour of pair-CSs (for the asymptotic formula of hypergeometrical function, see [13]). The possible situations are presented in table 1.

At the same time, the probability that the *n* excitations of mode (part) *a* and n + q excitations of mode (part) *b* will be found in the pair-CS $|z; q\rangle$ is

$$P_n^{|z;q\rangle} = |\langle n, n+q|z;q\rangle|^2 = \frac{1}{F} \frac{(|z|^2)^n}{\Gamma(n+1)\frac{\Gamma(2k+n)}{\Gamma(2k)}\frac{\Gamma(q+1+n)}{\Gamma(q+1)}\frac{\Gamma(q+2k+n)}{\Gamma(q+2k)}}.$$
 (28)

By using the asymptotic formula for the hypergeometric function [13], this probability distribution function can be split, up to an elementary function, into four generalized Gamma probability functions depending on the variable $\sqrt{|z|}$ with unitary scale parameter:

$$P_{n}^{|z;q\rangle} = 2 \left(2\pi\sqrt{|z|}\right)^{3/2} \frac{\left(\sqrt{|z|}\right)^{n} e^{-\sqrt{|z|}}}{\Gamma(n+1)} \frac{\left(\sqrt{|z|}\right)^{2k+n-1} e^{-\sqrt{|z|}}}{\Gamma(2k+n)}$$
$$\times \frac{\left(\sqrt{|z|}\right)^{q+n} e^{-\sqrt{|z|}}}{\Gamma(q+n+1)} \frac{\left(\sqrt{|z|}\right)^{q+2k+n-1} e^{-\sqrt{|z|}}}{\Gamma(q+2k+n)}$$
$$\equiv 2 \left(2\pi\sqrt{|z|}\right)^{3/2} P_{n+1}^{|z;0\rangle} P_{2k+n}^{|z;0\rangle} P_{q+1+n}^{|z;q\rangle} P_{q+2k+n}^{|z;q\rangle}, \qquad (29)$$

where the first two probabilities (being independent of q) are referred to subsystem a, the last two being connected with subsystem b.

4. Thermal properties of the pair-CSs

If we consider that the whole quantum system (a+b) obeys the canonical distribution, then a mixed state, in which both individual parts have an equal probability distribution function, is characterized by the following density operator:

$$\rho^{(q)} = \frac{1}{Z^{(q)}} \sum_{n=0}^{\infty} e^{-\beta(E_n + E_{n+q})} |n, n+q; k\rangle \langle n, n+q; k|.$$
(30)

The diagonal expansion of the density operator in terms of the pair-CSs

$$\rho^{(q)} = \int \mathrm{d}\mu(z;q) \, |z;q\rangle P(|z|^2;q)\langle z;q| \qquad (31)$$

is fulfilled if the *P* quasi-distribution function is expressed in terms of Meijer's G-functions as

$$P(|z|^{2};q) = \frac{e^{-\beta \left(E_{0}^{(q)}-2\hbar\omega\right)}}{Z^{(q)}} \times \frac{G_{04}^{40}\left(e^{\beta 2\hbar\omega}|z|^{2}\mid 0, 2k-1, q, 2k-1+q\right)}{G_{04}^{40}\left(|z|^{2}\mid 0, 2k-1, q, 2k-1+q\right)}.$$
(32)

5. Even- and odd-pair-CSs

Generally, the even and odd CSs are useful in quantum information theory, to use the pair-CSs as an entangled resource [14, 15]. For the PHO these pair-CSs can be constructed if we use the pair-CSs with the negative complex variable:

$$|-z;q\rangle = \frac{1}{\sqrt{F}} \sum_{n=0}^{\infty} \frac{(-z)^n}{\sqrt{\rho(n;k;q)}} |n, n+q;k\rangle.$$
 (33)

The overlap between the pair-CSs with positive and negative complex variable z is symmetric: $\langle -z; q | z; q \rangle = \langle z; q | -z; q \rangle$. Then, the balanced even (+) and odd (-) pair-CSs for the PHO are defined as follows:

$$|z;q\rangle^{(\pm)} = N^{(\pm)} \left(|z;q\rangle \pm |-z;q\rangle\right).$$
(34)

The normalization function $N^{(\pm)}$ is obtained from the condition ${}^{(\pm)}\langle z; q|z; q\rangle^{(\pm)} = 1$.

To calculate the Mandel parameter for the whole number operator $N^{(ab)} = N^{(a)} + N^{(b)}$ for the even and odd pair-CSs for PHO, we observe the following symmetries:

$$\langle z; q | [N^{(ab)}]^{s} | -z; q \rangle = \langle -z; q | [N^{(ab)}]^{s} | z; q \rangle$$

$$\equiv \langle [N^{(ab)}]^{s} \rangle_{z,-z;q} ,$$

$$\langle z; q | [N^{(ab)}]^{s} | z; q \rangle = \langle -z; q | [N^{(ab)}]^{s} | -z; q \rangle$$

$$\equiv \langle [N^{(ab)}]^{s} \rangle_{z,z;q} ,$$

$$(35)$$

$${}^{(\pm)}\langle z; q | [N^{(ab)}]^{s} | z; q \rangle^{(\pm)} \equiv \left\langle [N^{(ab)}]^{s} \right\rangle_{z,z;q}^{(\pm)}$$
$$= 2 [N^{(\pm)}]^{2} \left[\left\langle [N^{(ab)}]^{s} \right\rangle_{z,z;q} \pm \left\langle [N^{(ab)}]^{s} \right\rangle_{z,-z;q} \right].$$
(36)

The Mandel parameter for the even and odd pair-CSs becomes

$$Q_{z;q}^{(\pm)} = \frac{\langle N^2 \rangle_{z,z;q}^{(\pm)} - \left(\langle N \rangle_{z,z;q}^{(\pm)} \right)^2}{\langle N \rangle_{z,z;q}^{(\pm)}} - \langle N \rangle_{z,z;q}^{(\pm)}.$$
 (37)

Even if their expression seems to be intricate, it can provide some useful information about the statistical properties of even and odd pair-CSs of the PHO.

6. Concluding remarks

The PHO has a linear energy vibrational spectrum and, like the HO-1D, admits the construction of three kinds of CSs. In the present paper, we have constructed the Barut–Girardello pair-CSs for the PHO. We have examined the nonclassical properties of these pair-CSs, by calculating the corresponding Mandel parameter and also by using the density operator formalism: both for pure states (a pair-CSs projector) and for mixed (thermal) states. Generally, the CSs approach not only greatly simplifies the calculations of various expectation values for the examined quantum system, but also may be of potential use in developing the quantum information theory. Particularly, the pair-CSs are important in quantum information processing (as an entanglement resource or as a quantum transmission channel [14–16]).

The main difference between the pair-CSs and the pair of CSs lies in the property of entanglement of the pair-CSs. Namely, it is evident that a pair-CS cannot be decomposed into a tensorial product of two usual Barut–Girardello CSs [3], one referring to subsystem *a* and another to subsystem *b*. As the entanglement resource, the pair-CSs can be used, for example, in the quantum teleportation protocol analysed in [15] (not reproduced here due to paper length requirements). Namely, in this protocol, initially Alice and Bob share a state that is just a pair-CS. Consequently, Alice makes a joint measurement of the target state (that will be sent to Bob) and her component of received pair-CS and the result of joint measurement will be sent to Bob via the classical channel. Moreover, the pair-CSs can always be distillable in a phase-damping channel, so they are useful in quantum information processing.

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4